I. HILBERT SPACE

Definition 1. *Let X be a complex vector space, (thus X is closed under addition and complex scalar multiplication), a function $\langle \cdot, \cdot \rangle : X \times X \to C$ is called an* **inner product** *on X, if the following properties are satisfied:*

(i)Positivity: $\langle u, u \rangle > 0$ for each nonzero $u \in X$;

(ii)Conjugate symmetry: $\overline{\langle u, v \rangle} = \langle v, u \rangle$ for all $u, v \in X$;

(iii)Homogeneity: $\langle cv, u \rangle = c\langle v, u \rangle$ for all $u, v \in X$ and scalar $c \in C$;

(iv) Additivity: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in X$.

Example The space of all square integrable functions defined on the the interval $[-\pi, \pi]$ is usually denoted as $L^2([-\pi, \pi])$. More specifically, we denote

$$L^2([-\pi, \pi]) = \{f : [-\pi, \pi] \to \mathbb{C} \mid \int_{-\pi}^{\pi} |f(t)|^2 dt < \infty\}.$$

We can define inner product on it as

$$\langle f, g \rangle_{L^2([-\pi,\pi])} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)} dt$$

for $f, g \in L^2([-\pi, \pi])$.

Remark By Hölder's inequality, for any $f, g \in L^2([-\pi, \pi])$, we have

$$(\int_{-\pi}^{\pi} |f(t)g(t)|dt)^2 \leq \int_{-\pi}^{\pi} |f(t)|^2 dt \int_{-\pi}^{\pi} |g(t)|^2 dt,$$

Hence the inner product defined above makes sense.

In general, if $\langle \cdot, \cdot \rangle : X \times X \to C$ is an inner product defined on X,then

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle.$$

This is usually called **Cauchy-Schwartz Inequality**.

Definition 2. *Let X be a complex vector space on which an inner product $\langle \cdot, \cdot \rangle : X \times X \to C$ is defined (such spaces are called inner-product spaces). The* **norm** *$|| \cdot ||$ on X induced by the inner product can define as $||x|| = \langle x, x \rangle^2$. It can be checked using the properties of inner product and Cauchy-Schwartz Inequality that $|| \cdot ||$ satisfies the following*

(i)$||x|| = 0$ if and only if $x = 0$ for all $x \in X$;

(ii)$||cx|| = |c| \cdot ||x||$ for all $x \in X$ and $c \in C$;

(iii)$||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$;

Remark The last property above is usually called **Triangle Inequality**.

The properties (i),(iii) enjoyed by the norm remind one about *absolute value* of real numbers. Indeed, the "distance" between two elements in X can be measured by the norm of their difference. In particular, we can talk about Cauchy sequence and convergence of a sequence of elements $\{x_n\}_{n \in N} \subset X$.

Definition 3. . *Let X be a complex vector space on which an inner product is defined. Let $|| \cdot ||$ be the norm on X induced by the inner product. We say that a sequence of elements $\{x_n\}_{n \in N} \subset X$* **converges** *to $x \in X$ under the norm $|| \cdot ||$, if the sequence $\{||x_n - x||\}_{n \in N}$ of real numbers converges to 0, i.e., for any $\epsilon > 0$, there is $N \in \mathbb{N}$, such that for any natural number $n > N$, we have $||x_n - x|| < \epsilon$.*

Likewise, We say that the series $\sum_{n=1}^{\infty} x_n$ **converges** in X (or summable), if its partial sum sequence $\{\sum_{i=1}^{n} x_i\}_{n \in N}$ converges to some $x \in X$.

Remark It can be checked that whenever a sequence of elements $\{x_n\}_{n \in N} \subset X$ converges to $x \in X$ under the norm $|| \cdot ||$, then $\{x_n\}_{n \in N}$ satisfies the following property:

For any $\epsilon > 0$, there is $N \in \mathbb{N}$, such that for any natural number $m, n > N$, we have $||x_n - x_m|| < \epsilon$.

Any $\{x_n\}_{n \in N} \in X$ satisfying the the property above is called a **Cauchy Sequence** in X under the norm $|| \cdot ||$.

Whenever there is no ambiguity about norm we refer to in the context, we usually omit the phrase "under the norm $|| \cdot ||$".

Now we are ready to introduce the notion of a **Hilbert Space**.

Definition 4. . *Let X be a complex vector space on which an inner product is defined. Let $|| \cdot ||$ be the norm on X induced by the inner product. We say that X is a (complex)* **Hilbert space** *if every Cauchy sequence in X converges to some $x \in X$.*

Example The space $L^2([-\pi,\pi])$ mentioned above is a Hilbert space. The norm induced by the inner product $\langle f,g\rangle_{L^2([-\pi,\pi])}$ is

$$||f||_{L^2([-\pi,\pi])} = \frac{1}{\sqrt{2\pi}}(\int_{-\pi}^{\pi} |f(t)|^2 dt)^{\frac{1}{2}}$$

for all $f \in L^2([-\pi,\pi])$.

Example We use $\mathbb{Z}$ to denote the set of all integers. Define

$$l^2(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} \mid \sum_{k\in\mathbb{Z}} |f(k)|^2 < \infty\}.$$

If the inner product on $l^2(\mathbb{Z})$ is defined as

$$\langle f,g\rangle_{l^2(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} f(k)\overline{g(k)},$$

then $l^2(\mathbb{Z})$ becomes a Hilbert space (with the norm induced by the above defined inner product).

More often (and much more conveniently), the Hilbert space $l^2(\mathbb{Z})$ is represented as a space of "sequences":

$$l^2(\mathbb{Z}) = \{\{a_k\}_{k\in\mathbb{Z}} \subset \mathbb{C} \mid \sum_{k\in\mathbb{Z}} |a_k|^2 < \infty\}.$$

With such a representation, the inner product defined above has the following form:

For any elements $a = \{a_k\}_{k\in\mathbb{Z}}, b = \{b_k\}_{k\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$,

$$\langle a,b\rangle_{l^2(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} a_k\overline{b_k}.$$

In some sense, $l^2(\mathbb{Z})$ is the simplest among all infinite dimensional Hilbert Space. We will try to show how Hilbert spaces in general, $L^2([-\pi,\pi])$ in particular, is related to $l^2(\mathbb{Z})$. Let us introduce the notion of *orthogonality* first. (the rest of this section is mainly taken from Katznelson's An introduction to Harmonic Analysis.)

Definition 5. *Let $\mathcal{H}$ be a complex Hilbert space. Let $f,g \in \mathcal{H}$. We say that f is* **orthogonal** *to g if $\langle f,g\rangle = 0$. This relation is clearly symmetric. If E is a subset of $\mathcal{H}$, we say that $f \in \mathcal{H}$ is* **orthogonal** *to E if f is orthogonal to every element in E. A set $E \subset \mathcal{H}$ is* **orthogonal** *if any two vectors in E are orthogonal to each other. A set $E \subset \mathcal{H}$ is an* **orthonormal system** *if it is orthogonal and the norm* *of each vector in E is one, that is, if, whenever $f, g \in E$, $\langle f, g\rangle = 0$ if $f \neq g$ and $\langle f, f\rangle = 1$.*

Lemma 1. *Let $\{\varphi_n\}_{n=1}^N$ be a finite orthonormal system in Hilbert space $\mathcal{H}$. Let $a_1, ..., a_N$ be complex numbers. Then*

$$||\sum_{n=1}^N a_n\varphi_n||^2 = \sum_{n=1}^N |a_n|^2.$$

Proof.

$$||\sum_{n=1}^N a_n\varphi_n||^2 = \langle\sum_{n=1}^N a_n\varphi_n, \sum_{n=1}^N a_n\varphi_n\rangle = \sum_{n=1}^N a_n\langle\varphi_n, \sum_{n=1}^N a_m\varphi_m\rangle$$

$$= \sum_{n=1}^N a_n\overline{a}_n = \sum_{n=1}^N |a_n|^2$$

□

Corollary 1. *Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal system in Hilbert space $\mathcal{H}$. Let $\{a_n\}_n^\infty$ be a sequence of complex numbers such that $\sum_n^\infty |a_n|^2 < \infty$. Then $\sum_{n=1}^\infty a_n\varphi_n$ converges in $\mathcal{H}$.*

Proof. Since $\mathcal{H}$ is a Hilbert space, every Cauchy sequence in $\mathcal{H}$ converges to some vector in $\mathcal{H}$. Therefore all we have to show is that the partial sums $S_N = \sum_{n=1}^N a_n\varphi_n$ form a Cauchy sequence in $\mathcal{H}$.

Now, by Lemma 1, for $N > M$,

$$||S_N - S_M||^2 = ||\sum_{n=M+1}^N a_n\varphi_n||^2 = \sum_{n=M+1}^N |a_n|^2.$$

The rest is routine.

□

Lemma 2. *Let $\{\varphi_n\}_{n=1}^N$ be a finite orthonormal system in Hilbert space $\mathcal{H}$. For $f \in \mathcal{H}$, let $a_n = \langle f, \varphi_n\rangle$. Then*

$$0 \leq ||f - \sum_{n=1}^N a_n\varphi_n||^2 = ||f||^2 - \sum_{n=1}^N |a_n|^2.$$

Proof.

$$||f - \sum_{n=1}^N a_n\varphi_n||^2 = \langle f - \sum_{n=1}^N a_n\varphi_n, f - \sum_{n=1}^N a_n\varphi_n$$

$$= ||f||^2 - \sum_{n=1}^{N} \overline{a}_n \langle f, \varphi_n \rangle - \sum_{n=1}^{N} a_n \langle \varphi_n, f \rangle + \sum_{n=1}^{N} |a_n|^2 = ||f||^2 - \sum_{n=1}^{N} |a_n|^2.$$

□

Corollary 2. Bessel's Inequality *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system in Hilbert space $\mathcal{H}$. Then for any $f \in \mathcal{H}$, we have*

$$\sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 \leq ||f||^2.$$

Bessel's inequalty is an easy consequence of Lemma 2, we omit the proof. For any orthonormal system$\{\varphi_n\}_{n=1}^{\infty}$ in Hilbert space $\mathcal{H}$ and any sequence of complex numbers $\{a_n\}_n^{\infty}$, when $\sum_{n=1}^{\infty} a_n \varphi_n$ converges in $\mathcal{H}$ to some $f \in \mathcal{H}$, namely, the partial sum sequence $\{\sum_{i=1}^{n} a_i \varphi_i\}_{n=1}^{\infty}$ converges to $f \in \mathcal{H}$, we write

$$f = \sum_{n=1}^{\infty} a_n \varphi_n.$$

Next we show an important consequence of Cauchy-Schwartz inequality.

Lemma 3. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system in Hilbert space $\mathcal{H}$, $\{a_n\}_n^{\infty}$ be a sequence of complex numbers $\{a_n\}_n^{\infty}$. If $f = \sum_{n=1}^{\infty} a_n \varphi_n$, then for any $g \in \mathcal{H}$, we have*

$$\langle f, g \rangle = \sum_{n=1}^{\infty} a_n \langle \varphi_n, g \rangle.$$

Proof. We need to show that the series of complex numbers $\sum_{n=1}^{\infty} a_n \langle \varphi_n, g \rangle$ converges to the complex number $\langle f, g \rangle$.

Now

$$| \langle f, g \rangle - \sum_{n=1}^{N} a_n \langle \varphi_n, g \rangle | = | \langle f - \sum_{n=1}^{N} a_n \varphi_n, g \rangle | \leq ||f - \sum_{n=1}^{N} a_n \varphi_n||^{\frac{1}{2}} ||g||^{\frac{1}{2}}$$

where the last inequality is Cauchy-Schwartz inequality.

The rest of the proof is routine. □

Definition 6. *A* **complete orthonormal system***in $\mathcal{H}$ is an orthonormal system having the additional property that the only vector in $\mathcal{H}$ orthogonal to it is the zero vector.*

Lemma 4. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system in Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(a) $\{\varphi_n\}_{n=1}^{\infty}$ is complete.

(b) For every $f \in \mathcal{H}$ we have

$$||f||^2 = \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2.$$

(c) For every $f \in \mathcal{H}$ we have

$$f = \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle \varphi_n.$$

Proof.

To prove (b)$\Longleftrightarrow$(c), we only need to look at Lemma 2. Details are left to the reader.

To prove (b)$\Longrightarrow$ (a), assuming vector f is orthogonal to $\{\varphi_n\}_{n=1}^{\infty}$, then by (b) we see that $||f|| = 0$ which means $f = 0$, hence $\{\varphi_n\}_{n=1}^{\infty}$ is complete.

Lastly, we show (a)$\Longrightarrow$(c).

Since $\sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 \leq ||f||^2$ by Bessel's inequality, according to Corollary 1, we see that $\sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$ converges in $\mathcal{H}$. Let us denote $g = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$, we are done if we can show $f = g$. Indeed, using Lemma 3, we compute to get $\langle g, \varphi_n \rangle = \langle f, \varphi_n \rangle$ for each $n \in \mathbb{N}$. This means $\langle g - f, \varphi_n \rangle = 0$ for each $n \in \mathbb{N}$. Since $\{\varphi_n\}_{n=1}^{\infty}$ is complete, it follows from definition that $g - f = 0$.

□

Corollary 3. Parseval's Identity *Let $\{\varphi_n\}_{n=1}^{\infty}$ be acomplete orthonormal system in Hilbert space $\mathcal{H}$, let $f, g \in \mathcal{H}$. Then*

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \langle \varphi_n, g \rangle.$$

Parseval's Identity is an easy consequence of Lemma 3 and Lemma 4, we leave the proof for the reader. Now we turn to the concrete Hilbert space $L^2([-\pi, \pi])$. Consider $\{e^{int}\}_{n \in \mathbb{Z}} \subset L^2([-\pi, \pi])$, since

$$\langle e^{int}, e^{imt} \rangle_{L^2([-\pi,\pi])} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} \overline{e^{imt}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)t} dt = \delta_{n,m}.$$

clearly $\{e^{int}\}_{n \in \mathbb{Z}}$ is an orthonormal system in $L^2([-\pi, \pi])$. In fact it is also complete, but the proof of its completeness is beyond the scope of this course. The general results about complete orthonormal systems in Hilbert space now yield

Theorem 1. *For any $f \in L^2([-\pi,\pi])$, denote $\langle f, e^{int}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{e^{imt}}dt = \hat{f}(n)$ for each $n \in \mathbb{Z}$. Then*

(a)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

(b)

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}$$

where the convergence is in the norm of $L^2([-\pi,\pi])$.

(c) Given any sequence $\{a_n\}_{n\in\mathbb{Z}}$ satisfying $\sum_{n\in F} |a_n|^2 < \infty$, there is a unique $f \in L^2([-\pi,\pi])$ such that $a_n = \hat{f}(n)$ for each $n \in \mathbb{Z}$.

(d) For any $f, g \in L^2([-\pi,\pi])$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}.$$

This brings the saga to an end.

The proof is mostly trivial. We omit the proof, only to mention that to prove (c), by Corollary 1 we can first write $f = \sum_{n\in\mathbb{Z}} a_n e^{int}$, then by using Lemma 3 and the fact that $\{e^{int}\}_{n\in\mathbb{Z}}$ is complete, we can prove the uniqueness part.

Remark The above theorem shows that $L^2([-\pi,\pi])$ and $l^2(\mathbb{Z})$ can be "identified" in certain sense through the correspondence $f \longleftrightarrow \{\hat{f}(n)\}$, mathematically it is called an **isometry** but we will not get into the that.